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# Modulational instability arising from collective Rayleigh scattering

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It is shown that under certain conditions a collection of dielectric Rayleigh particles suspended in a viscous medium and enclosed in a bidirectional ring cavity pumped by a strong laser field can produce a new modulational instability transverse to the wave-propagation direction. The source of the instability is collective Rayleigh scattering i.e., the spontaneous formation of periodic *longitudinal* particle-density modulations and a backscattered optical field. Using a linear stability analysis a dispersion relation is derived which determines the region of parameter space in which modulational instability of the backscattered field and the particle distribution occurs. In the linear regime the pump is modulationally stable. A numerical analysis is carried out to observe the dynamics of the interaction in the nonlinear regime. In the nonlinear regime the pump field also becomes modulationally unstable and strong pump depletion occurs.

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## I. INTRODUCTION

Modulational or filamentation instability is a fundamental nonlinear wave-propagation phenomenon which has been extensively studied in a wide range of physical systems including fluids [1], plasmas [2], nonlinear optical systems [3], and Bose-Einstein condensates [4]. The main feature of modulational instability is the breakup of initially uniform wavefronts into localized structures.

In this paper we predict the existence and evolution of a new modulational instability occurring in a suspension of dielectric Rayleigh particles enclosed in a ring cavity illuminated by laser light. We first give a brief description of the mean-field model we use to describe this phenomenon. We then show that scattering of the illuminating (pump) laser light by the particles gives rise to a collective Rayleigh scattering instability. This instability is similar to those which occur in the high-gain free electron laser [5] and the collective atomic recoil laser [6]. It produces a simultaneous amplification of a backscattered (probe) radiation field and a longitudinal periodic density modulation in the particle distribution with a period approximately half the wavelength of the pump. Using a linear stability analysis we show that it can also produce a modulational instability of the probe field and the particle distribution transverse to the wave-propagation direction, but that the pump field is modulationally stable in the linear regime. Finally we use a numerical analysis to describe the nonlinear evolution of the interaction. We find that in this regime the modulational instability which has developed in the probe field and particle distribution is transferred to the pump field.

## II. MODEL

In this section we derive the evolution equations which describe the interaction between an ensemble of dielectric particles suspended in a viscous medium and two counter-propagating modes of a wide-aperture bidirectional ring cavity, as shown schematically in Fig. 1. We use a mean-field and one-dimensional approximation to describe optical fields in the cavity, assuming that only one mode is excited in each direction of propagation, and that the field envelopes have only one transverse degree of freedom $(x)$. The form of the $\mathbf{E}$ field in the cavity is therefore,

$$\mathbf{E}=[A_1(x,t)e^{i(k_c z-\omega_c t)}+A_2(x,t)e^{-i(k_c z+\omega_c t)}+\text{c.c.}]\hat{\mathbf{x}}, \tag{1}$$

where $A_{1,2}$ are complex field amplitudes which we refer to as probe (1) and pump (2), respectively, $k_c=2\pi m/\mathcal{L}$ is the mode wave number, $\mathcal{L}$ is the cavity length, $m$ is an integer, $\omega_c=(k_c c/\sqrt{\epsilon_m^{\text{eff}}})$ is the mode (angular) frequency, $\epsilon_m^{\text{eff}}=(n_m^{\text{eff}})^2=1+(\chi_m L/\mathcal{L})$ is the effective relative permittivity of the partially filled cavity, $\chi_m$ is the (real) susceptibility of the viscous medium, and $\hat{\mathbf{x}}$ is a transverse unit vector. For simplicity we have assumed that the pump and probe are frequency degenerate.

The force on the $j$th particle exerted by the optical cavity fields can be derived from the Lorentz force equation to be

$$\mathbf{F}_j=\partial\mathbf{d}_j/\partial t\times\mathbf{B}(z_j(t)), \tag{2}$$

where $\mathbf{B}(x,z,t)=(n_m^{\text{eff}}/c)\hat{\mathbf{z}}\times(\mathbf{E}_1-\mathbf{E}_2)$ is the magnetic field of the electromagnetic wave, $\mathbf{d}_j$ is the dipole moment of the $j$th particle induced by the electric field at $z=z_j$ (the axial position of the $j$th particle) given by

$$\mathbf{d}_j=\epsilon_0\epsilon_m^{\text{eff}}V_p[\chi(A_1(x_j,t)e^{i(k_c z_j-\omega_c t)}+A_2(x_j,t)e^{-i(k_c z_j+\omega_c t)})+\text{c.c.}]\hat{\mathbf{x}},$$

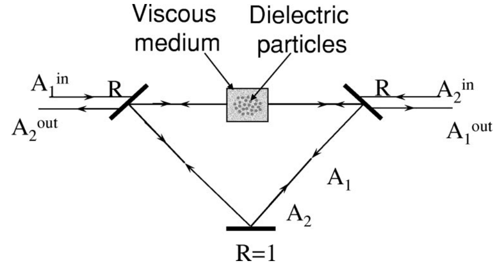


FIG. 1. Schematic diagram of a suspension of dielectric particles enclosed in a wide-aperture bidirectional ring cavity.

 

where $\epsilon_0$ is the permittivity of free space, $V_p = 4\pi a^3/3$ is the particle volume, and $a$ is the particle radius. The susceptibility of the dielectric particle is $\chi = \chi_1 + i\chi_2$, where

$$\chi_1 = 3\left(\frac{\epsilon_p/\epsilon_m^{\mathrm{eff}} - 1}{\epsilon_p/\epsilon_m^{\mathrm{eff}} + 2}\right) \text{ and } \chi_2 = 2(k_c a)^3\left(\frac{\epsilon_p/\epsilon_m^{\mathrm{eff}} - 1}{\epsilon_p/\epsilon_m^{\mathrm{eff}} + 2}\right)^2$$

represent the dispersive and dissipative response of the dipole, respectively, $\epsilon_p$ is the relative permittivity of each particle. The dissipative response is due to damping via re-radiation, i.e., incoherent Rayleigh scattering. It was shown in Ref. [7] that the dynamics of the particles under the influence of the electromagnetic fields, the viscous drag force due to the medium, and the stochastic Brownian forces exerted on the particle by the molecules of the suspending medium can be described by a Fokker-Planck equation describing the evolution of the particle probability distribution. The dipole moments of the moving particles constitute a time-dependent polarization which drives the evolution of the probe and pump radiation fields. The dynamics of the particles and the fields are simultaneously described by a set of coupled Maxwell-Fokker-Planck equations [7], modified to take account of the presence of the ring cavity [8] and the transverse evolution of the fields:

$$\frac{\partial P(\bar{x},\theta,\bar{t})}{\partial \bar{t}} = -\frac{\partial}{\partial\theta}\left[\left(-\frac{1}{\bar{\gamma}}(\bar{A}_1\bar{A}_2^* e^{i\theta} + \bar{A}_1^*\bar{A}_2 e^{-i\theta}) + \frac{\alpha}{\bar{\gamma}}(|\bar{A}_1|^2 - |\bar{A}_2|^2)\right)P(\bar{x},\theta,\bar{t})\right] + \overline{\gamma\sigma^2}\frac{\partial^2 P(\bar{x},\theta,\bar{t})}{\partial\theta^2}, \tag{3}$$

$$\frac{\partial \bar{A}_1(\bar{x},\bar{t})}{\partial \bar{t}} - i\frac{\partial^2 \bar{A}_1(\bar{x},\bar{t})}{\partial \bar{x}^2} = (1+i\alpha)\left(2\pi\bar{A}_2\int_0^{2\pi} P(\bar{x},\theta,\bar{t})e^{-i\theta}d\theta + i\bar{A}_1\right) + i\delta_c\bar{A}_1 - \kappa\left(\bar{A}_1 - \frac{\bar{A}_1^{\mathrm{in}}}{\sqrt{1-R}}\right), \tag{4}$$

$$\frac{\partial \bar{A}_2(\bar{x},\bar{t})}{\partial \bar{t}} - i\frac{\partial^2 \bar{A}_2(\bar{x},\bar{t})}{\partial \bar{x}^2} = (1+i\alpha)\left(-2\pi\bar{A}_1\int_0^{2\pi} P(\bar{x},\theta,\bar{t})e^{i\theta}d\theta + i\bar{A}_2\right) + i\delta_c\bar{A}_2 - \kappa\left(\bar{A}_2 - \frac{\bar{A}_2^{\mathrm{in}}}{\sqrt{1-R}}\right), \tag{5}$$

where $\bar{x} = 2k_c\sqrt{\rho}x$ and $\bar{t} = 2\omega_c\rho t$ are scaled position and time variables, $\bar{A}_{1,2}(\bar{t}) = -2i\sqrt{\epsilon_0\epsilon_m^{\mathrm{eff}}/\rho N_p M c^2}A_{1,2}(t)$ are scaled complex probe (1) and pump (2) envelopes, respectively, $\bar{A}_{1,2}^{\mathrm{in}}$ are scaled input field amplitudes, $\bar{\gamma} = 3\pi a\eta/\omega_c\rho M$ is a scaled viscous damping coefficent, $\bar{\sigma} = (n_m^{\mathrm{eff}}/\rho)\sqrt{k_B T/Mc^2}$ is a scaled temperature coefficient, $\alpha = \chi_2/\chi_1$ is an incoherent scattering coefficient, $\kappa = (1-R)/2k_c\rho\mathcal{L}$ is a scaled cavity linewidth, $\delta_c = (\omega - \omega_c)/2\omega_c\rho$ is the scaled field-cavity detuning, $\rho = N_p V_p \chi_1/4$ is a dimensionless coupling parameter, $\theta = 2k_c z$ is a dimensionless spatial variable and it has been assumed that the viscous damping is sufficiently strong that inertial effects are negligible [7]. $k_B$ is Boltzmann's constant, $T$ is temperature, $M$ is the mass of each particle, $N_p$ is the particle number density in the *cavity*, i.e., $N_p = N_s(L/\mathcal{L})$, where $L$ is the sample length, $N_s$ is the number density of the sample, and $\eta$ is the coefficent of viscosity for the medium.

Assuming $P(\bar{x},\theta,\bar{t})$ is periodic in $\theta$ with period $2\pi$, i.e., periodic in $z$ with period $\lambda_c/2$, where $\lambda_c$ is the wavelength of the cavity mode and pump field, the distribution function $P(\bar{x},\theta,\bar{t})$ can then be expressed in terms of its spatial harmonics $P_k(\bar{x},\bar{t})$ using

$$P(\theta,\bar{x},\bar{t}) = \sum_{k=-\infty}^{\infty} P_k(\bar{x},\bar{t})e^{ik\theta},$$

where $P_0 = 1/(2\pi)$ and $P_{-n} = P_n^*$. The set of Maxwell-Fokker-Planck equations (3)–(5) can then be written as [7]

$$\frac{\partial P_k(\bar{x},\bar{t})}{\partial \bar{t}} = i\frac{k}{\bar{\gamma}}(\bar{A}_1\bar{A}_2^* P_{k-1} + \bar{A}_1^*\bar{A}_2 P_{k+1}) - k\left(\frac{i}{\bar{\gamma}}\alpha(|\bar{A}_1|^2 - |\bar{A}_2|^2) + \overline{\gamma\sigma^2}\right)P_k, \tag{6}$$

$$\frac{\partial \bar{A}_1(\bar{x},\bar{t})}{\partial \bar{t}} - i\frac{\partial^2 \bar{A}_1(\bar{x},\bar{t})}{\partial \bar{x}^2} = (1+i\alpha)(2\pi\bar{A}_2 P_1 + i\bar{A}_1) + i\delta_c\bar{A}_1 - \kappa\left(\bar{A}_1 - \frac{\bar{A}_1^{\mathrm{in}}}{\sqrt{1-R}}\right), \tag{7}$$

$$\frac{\partial \bar{A}_2(\bar{t})}{\partial \bar{t}} - i\frac{\partial^2 \bar{A}_2(\bar{x},\bar{t})}{\partial \bar{x}^2} = (1+i\alpha)(-2\pi\bar{A}_1 P_1^* + i\bar{A}_2) + i\delta_c\bar{A}_2 - \kappa\left(\bar{A}_2 - \frac{\bar{A}_2^{\mathrm{in}}}{\sqrt{1-R}}\right). \tag{8}$$

Note that this model considers only longitudinal ponderomotive forces and neglects transverse ones. This is because the length scale of the longitudinal ponderomotive potential is $\approx\lambda_c/2$, which is the wavelength of the cavity mode. The length scales of any transverse inhomogeneities arising from modulational instability will be much larger than $\lambda_c$ and consequently transverse ponderomotive forces will be much weaker than longitudinal ones.

It can be seen from Eqs. (7) and (8) that a longitudinal particle-density modulation ($P_1$) can couple the pump and probe fields. The nature of the coupling depends on the amplitude and the phase of the density modulation, i.e., the position in the ponderomotive potential formed by the interference of the pump and probe around which the particles bunch. It is this density modulation on a short length scale $\approx\lambda_c/2$ parallel to the field propagation direction which gives rise to a modulational instability with a length scale $\gg\lambda_c$ in the plane *perpendicular* to the wave-propagation direction.

### III. LINEAR ANALYSIS

In this paper we will consider only cases where there is no input probe beam ($|\bar{A}_1^{\mathrm{in}}|=0$). In this situation, Eqs. (6)–(8) have a steady-state unidirectional, homogeneous solution

$$\bar{A}_1^{ss}=0, P_k^{ss}=0(k\neq 0), \quad \bar{A}_2^{ss}=\frac{\kappa\bar{A}_2^{\mathrm{in}}/\sqrt{1-R}}{\kappa-i\delta_c-i(1+i\alpha)}.$$

The linear stability of this steady state can be determined by considering small fluctuations in $P_k$, $\bar{A}_1$, and $\bar{A}_2$, i.e.,

$$\bar{A}_1=a_1(\bar{x},\bar{t}), \quad P_k=p_k(\bar{x},\bar{t}), \quad \bar{A}_2=\bar{A}_2^{ss}+a_2(\bar{x},\bar{t}).$$

Treating these fluctuations as first-order quantities and $\bar{A}_2^{ss}$ as a zeroth-order quantity, we can write Eqs. (6)–(8) to first order as

$$\frac{\partial p_1}{\partial \bar{t}}=\frac{i}{\bar{\gamma}}\frac{a_1\bar{A}_2^{ss*}}{2\pi}-\left(-i\frac{\alpha|\bar{A}_2^{ss}|^2}{\bar{\gamma}}+\bar{\gamma}\bar{\sigma}^2\right)p_1, \tag{9}$$

$$\frac{\partial a_1}{\partial \bar{t}}-i\frac{\partial^2 a_1}{\partial \bar{x}^2}=(1+i\alpha)(2\pi\bar{A}_2^{ss}p_1+ia_1)+i\delta_c a_1-\kappa a_1, \tag{10}$$

$$\frac{\partial a_2}{\partial \bar{t}}-i\frac{\partial^2 a_2}{\partial \bar{x}^2}=i(1+i\alpha)a_2+i\delta_c a_2-\kappa a_2. \tag{11}$$

It can be seen from Eq. (11) that the pump field decouples from the probe field $a_1$ and density modulation $p_1$. Assuming the spatial dependence of $a_2$ is $\propto\cos(q\bar{x})$, Eq. (11) can be solved to show that

$$a_2(\bar{t})\propto e^{i(1+\delta_c-q^2)\bar{t}}e^{-(\alpha+\kappa)\bar{t}},$$

so the pump field is stable. Assuming the same spatial dependence for $a_1$ and $p_1$ as was done for the pump, i.e., $a_1,p_1\propto\cos(q\bar{x})$, the coupled linear equations for $a_1$ and $p_1$, Eqs. (9) and (10), can be solved using Laplace transforms. It can be shown that $a_1$ and $p_1$ will grow exponentially in time, $\propto e^{\lambda t}$, if the dispersion relation

$$[\lambda-i(1+i\alpha)-i(\delta_c-q^2)+\kappa]\left(\lambda+\bar{\gamma}\bar{\sigma}^2-i\frac{\alpha|\bar{A}_2^{ss}|^2}{\bar{\gamma}}\right)$$
$$-i(1+i\alpha)\frac{|\bar{A}_2^{ss}|^2}{\bar{\gamma}}=0 \tag{12}$$

has roots with positive real parts (Re $(\lambda)>0$).

Figure 2 shows graphs of maximum scaled growth rate $\mathrm{Re}(\lambda)_{\max}$ and the scaled transverse wave number $q_{\max}$ at which this maximum growth rate occurs, against scaled field-cavity detuning $\delta_c$. The parameters used were $\bar{A}_2^{\mathrm{in}}=0.0315$, $\bar{\gamma}=6.25$, $\bar{\sigma}=5.49\times10^{-3}$, $\alpha=0$, $\kappa=0.081$, and $R=0.97$. Physical parameters thatat correspond to these scaled quantities are $\lambda_c\approx532$ nm, $N_s=10^{11}$ cm$^{-3}$, $L=1$ cm, $\mathcal{L}=10$ cm, $\epsilon_m=1.77$, $\epsilon=2.56$, $a=25$ nm, $T=300$ K, $\eta=1.002\times10^{-3}$ Pa s, $R=0.97$, and pump intensity $I_2=1.27\times10^8$ W cm$^{-2}$. Note that we have neglected incoherent scattering ($\alpha=0$) rather than use the value consistent with these physical parameters ($\alpha=0.006$). This does not change the main features of the interaction in either the linear or nonlinear regime, but makes it easier to highlight the features which are of most interest. The purpose of this communication is to identify and describe general features of the modulational instabiltility rather than to perform a detailed simulation of a particular experimental configuration. These parameters correspond to a collection of latex microspheres suspended in water. It can be seen that maximum growth rate occurs at $\delta_c=-1.03$ and $q=0$, where the probe field and particle-density modulation will grow with a transversely homogeneous profile. A region of modulational instability (maximum growth rate $q_{\max}>0$) can be seen for $\delta_c>-1.03$. Here the probe amplitude and particle-density modulation amplitude are modulationally unstable, growing exponentially in time and developing strong transverse modulations.

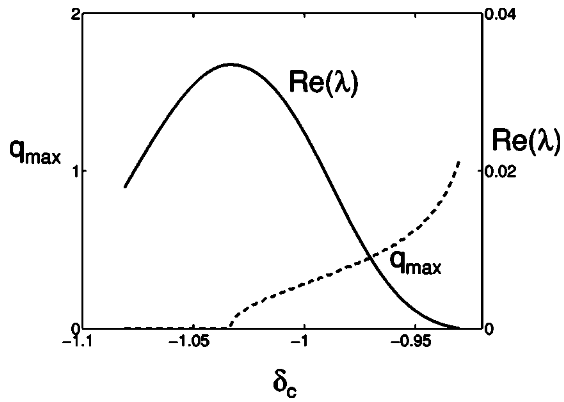


FIG. 2. Graph showing maximum scaled growth rate $\mathrm{Re}[\lambda(\delta_c,q)]$ and the scaled wave number $q_{\max}$ at which maximum growth occurs against scaled field-cavity detuning $\delta_c$, when $|\bar{A}_2^{\mathrm{in}}|=0.0315$, $\bar{\sigma}=5.49\times10^{-3}$, $\bar{\gamma}=6.25$, $\alpha=0$, $\kappa=0.081$, and $R=0.97$.

### IV. NONLINEAR ANALYSIS

In order to observe the evolution of the modulational instability predicted by the linear analysis into the nonlinear regime, we use a split-step Fourier method to solve Eqs. (6)–(8) numerically. The number of longitudinal spatial harmonics $k_{\max}$ used is chosen to be sufficiently large that the solution is unaffected by further increasing $k_{\max}$. The initial conditions used are $\bar{A}_1(\bar{x},\bar{t}=0)=\bar{A}_{1_0}r(\bar{x})$, $\bar{A}_2(\bar{x},\bar{t}=0)=\bar{A}_2^{\mathrm{in}}/(\sqrt{1-R})$, and $P_k(\bar{x},\bar{t}=0)=0$, where $r(\bar{x})$ is a random number in the range (0,1]. We have used a weak noisy ‘‘seed’’ probe field $\bar{A}_{1_0}$ to initiate the interaction. We could equally well have used no initial probe field and a small ‘‘seed’’ density perturbation $|P_1|\ll1$ to initiate the interaction and obtain essentially the same results. Figure 3(a) shows a graph of scaled intracavity probe intensity as a function of $\bar{x}$ and $\bar{t}$ as calculated from Eqs. (6)–(8) using the parameters $|\bar{A}_{10}|=1\times10^{-5}$, $\bar{A}_2^{\mathrm{in}}=0.0315$, $\bar{\gamma}=6.25$, $\bar{\sigma}=5.49\times10^{-3}$, $\delta_c=-0.95$, $\alpha=0$, $\kappa=0.081$, and $R=0.97$. The scaled transverse dimension of the window here is 50, which corresponds to a real transverse width of 5.2 mm. These parameters have been chosen so that the system lies in the region of instability calculated from the linear analysis in the preceding section. It can be seen from Fig. 3(a) that the scaled probe intensity rapidly develops a strong periodic transverse modulation as it is amplified. Figure 3(b) shows a similar graph of intracavity pump intensity. It can be seen from Fig. 3(b) that strong depletion of the pump field occurs in the nonlinear regime ($\bar{t} > 250$ approximately). This behavior of the intracavity fields is simultaneous with the amplification of $|P_1|$. Therefore, the probe amplification and strong pump depletion is due to the spontaneous formation of a periodic density modulation of the particle density with spatial period $\lambda_c/2$. In other words a one-dimensional optical bandgap has spontaneously formed in the medium of suspended particles [8]. Saturation of the probe intensity occurs after a scaled time of $\bar{t} \approx 300$, primarily due to strong depletion of the pump field. For these parameters, this corresponds to a real saturation time of $\approx 0.27 \mu$s. In addition to pump depletion, excitation of several higher longitudinal spatial harmonics of the particle-density distribution is also a feature of the nonlinear regime [8].

It can be seen that the pump field, while still modulationally stable in the linear regime, is now modulationally unstable in the nonlinear regime.

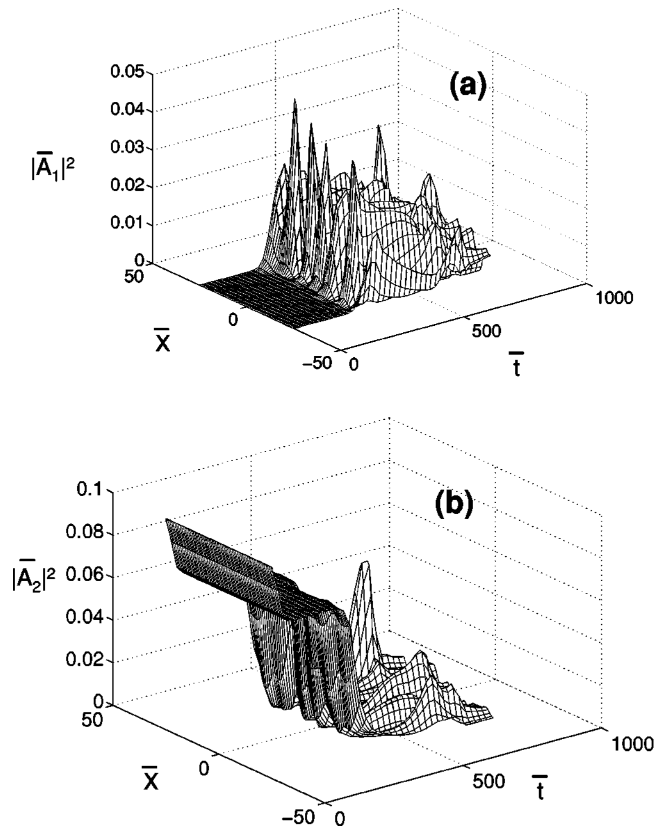


FIG. 3. Nonlinear evolution of (a) scaled intracavity probe intensity and (b) scaled intracavity pump intensity as a function of $\bar{x}$ and $\bar{t}$.

## V. CONCLUSIONS

It has been shown that under certain conditions a collection of dielectric Rayleigh particles suspended in a viscous medium and enclosed in a moderately high-$Q$ bidirectional ring cavity pumped by a strong laser field can give rise to a transverse modulational instability. Using a linear stability analysis a dispersion relation which determines the region of parameter space, in which the probe field and the particle distribution become modulationally unstable was derived. The pump field was found to be modulationally stable in the linear regime. A numerical solution of the evolution equations was carried out to observe the dynamics of the interaction in the nonlinear regime. The main features of the nonlinear evolution were strong pump depletion, as originally predicted in the plane-wave analysis of Ref. [8] and that the modulational instability which initially developed in the probe field and transverse particle distribution was subsequently transferred to the pump field.

The results reported here suggest that suspensions of dielectric particles may be useful media for the investigation of phenomena related to transverse modulational instability such as optical patterns, spatial solitons, and cavity solitons. A more detailed analysis of the one-dimensional model described here and its extension to include, e.g., two transverse dimensions will be carried out in a future extended publication. This analysis will also consider effects which could possibly exist simultaneously with the modulational instability reported here and may have to be taken into account in experimental attempts to observe the phenomena described in the manuscript, e.g., stimulated Brillouin scattering from the viscous medium.

## ACKNOWLEDGMENTS

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